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Fair division of multiple stochastic pies within the Nash bargaining solution: Modeling and real-life application

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Talk (12 Oct 2012, 17:00) "Theory Group" Department of Informatics, AUEB

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The model presented here is included in our recent paper:

- "On the fair division of multiple stochastic pies to multiple agents within the Nash bargaining solution", *PloS ONE*, Vol. 7, Issue 9, art. No. e44535

This paper as well as the Proofs of Propositions and Theorems presented here can be found:

http://dx.plos.org/10.1371/journal.pone.0044535

Topics:

- Introduction
- Problem Description and Mathematic Formulation
- Computing Solutions for Fair Surplus Division
- Basic Features
- Computation Algorithm
- A Real-life Application
- Discussion and Future Research

Cooperative game theory examines situations where at least two players benefit by working together or sharing some cost.

- Their objectives are partially cooperative, as they aim at reaching an agreement and partially conflicting, because each player has its own utility function regarding the negotiation outcome.

- Each player is faced with a set S of feasible bargaining outcomes, any of which presents the result if all participants agree upon.

A cooperative game for the coalition $N = \{1, 2\}$, is:

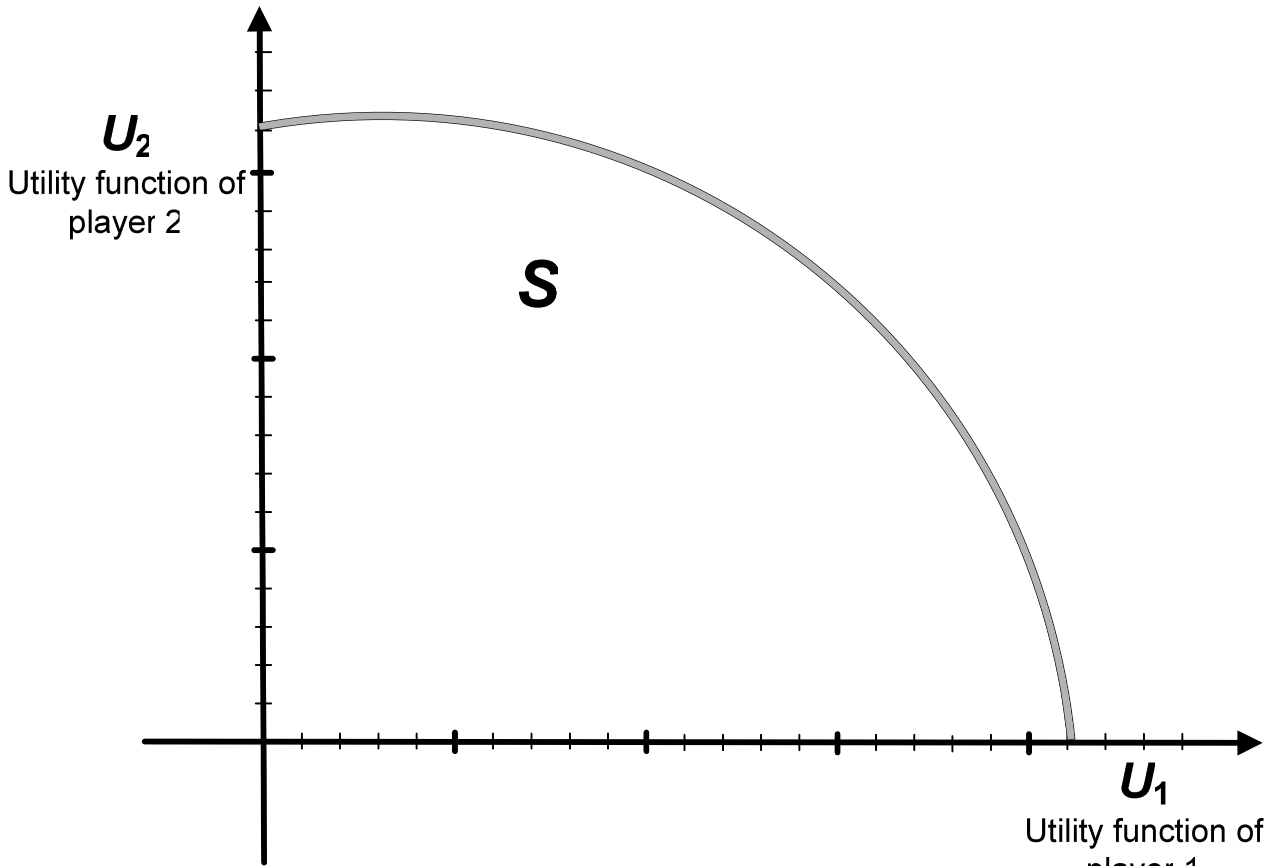
- Either a pair (N, p) with a characteristic function $p : 2^N \rightarrow \mathfrak{R}$ and $p(\varnothing) = 0$, which represents the collective payoff,

- Or a pair (N, c) and a characteristic function $c : 2^N \rightarrow \mathfrak{R}$ and $c(\varnothing) = 0$ that describes the cost for agents who cooperate in accomplishing a specific task.

The solution of the game is a vector $x \in \mathfrak{R}^N$ representing the allocation of the overall profit $p(N)$ or cost $c(N)$ to each agent.

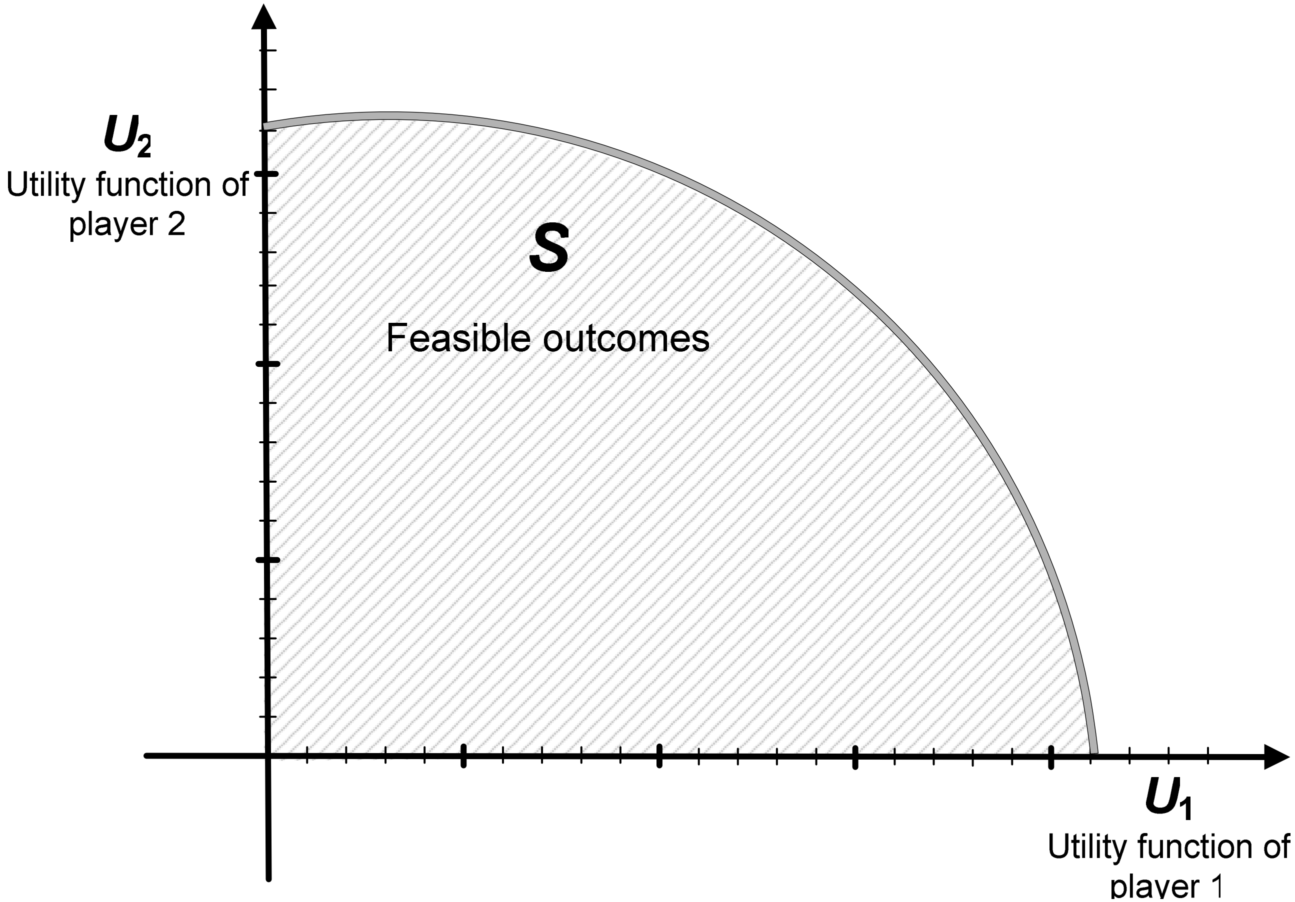
The Basic Problem: Two players (1 and 2) negotiate over a surplus that is yielded through cooperation.

- Each player has a utility function (von Neumann and Morgenstern, 1944).
- S is as a sub-set of $\Re^2$ (for n-players of $\Re^N$) and is assumed to be closed, convex, non-empty and bounded.

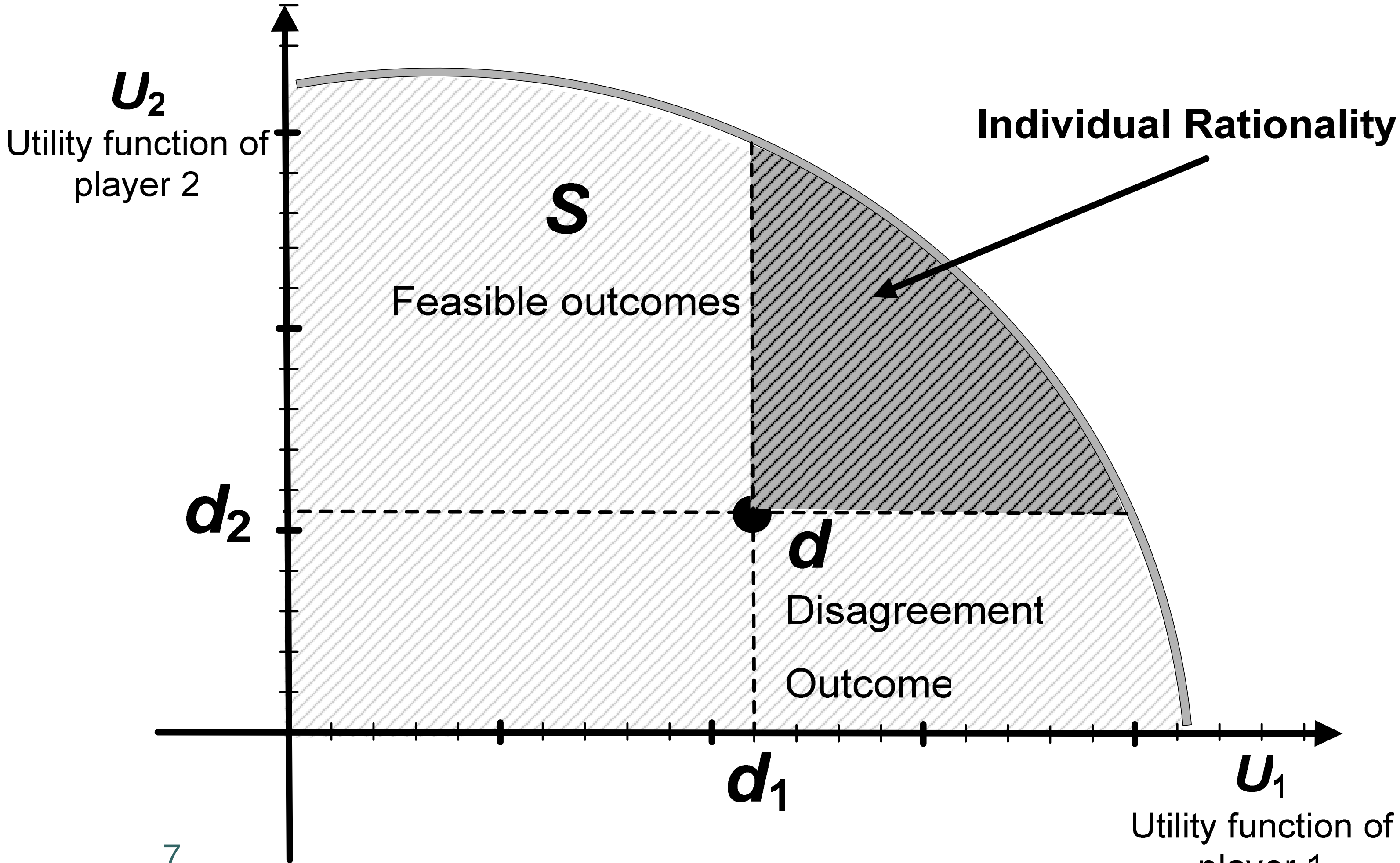


- Some natural axioms of the solution
- "One states as axioms several properties that it would be natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely" (Nash, 1953)

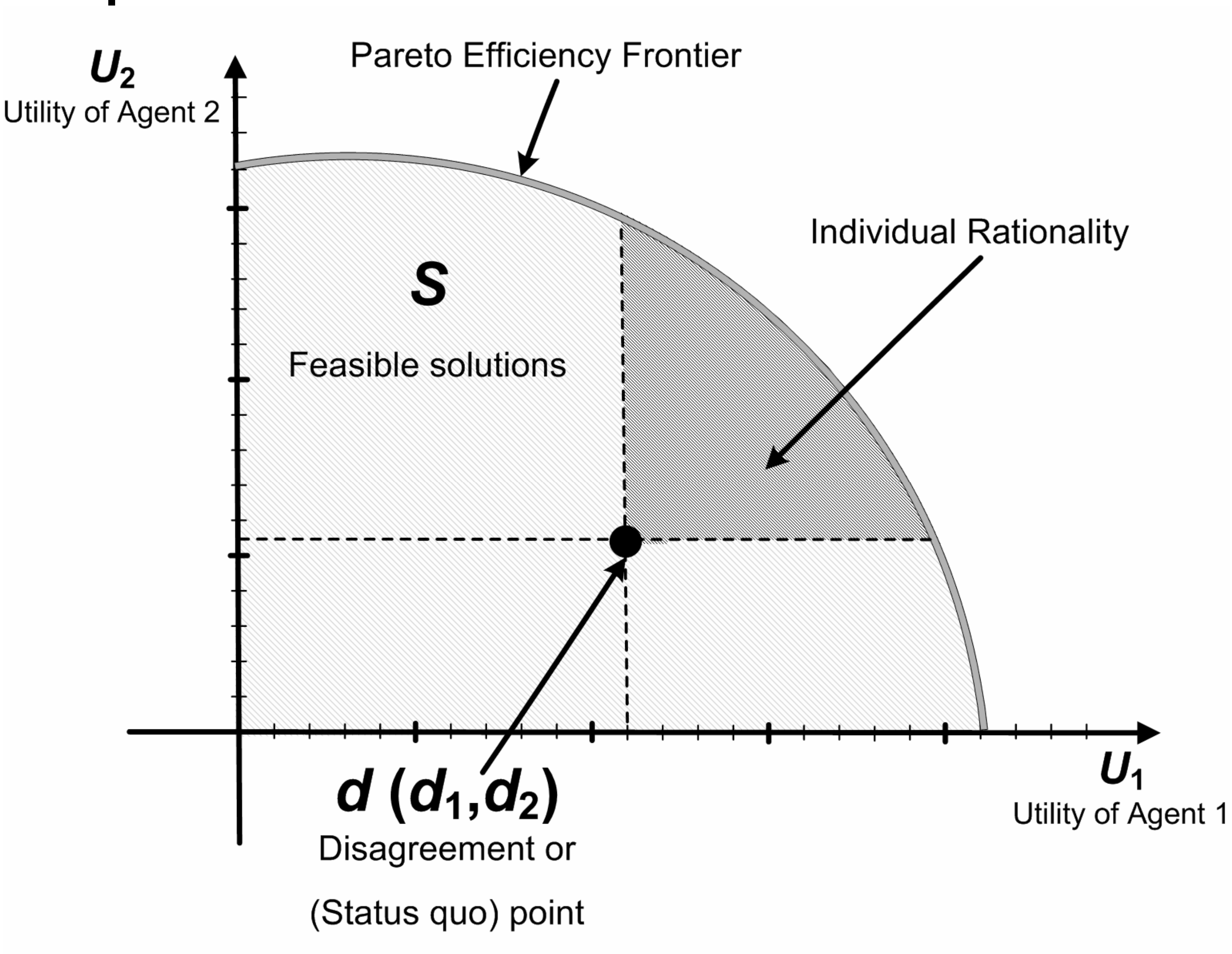
- **Axiom 1.** *Feasibility*



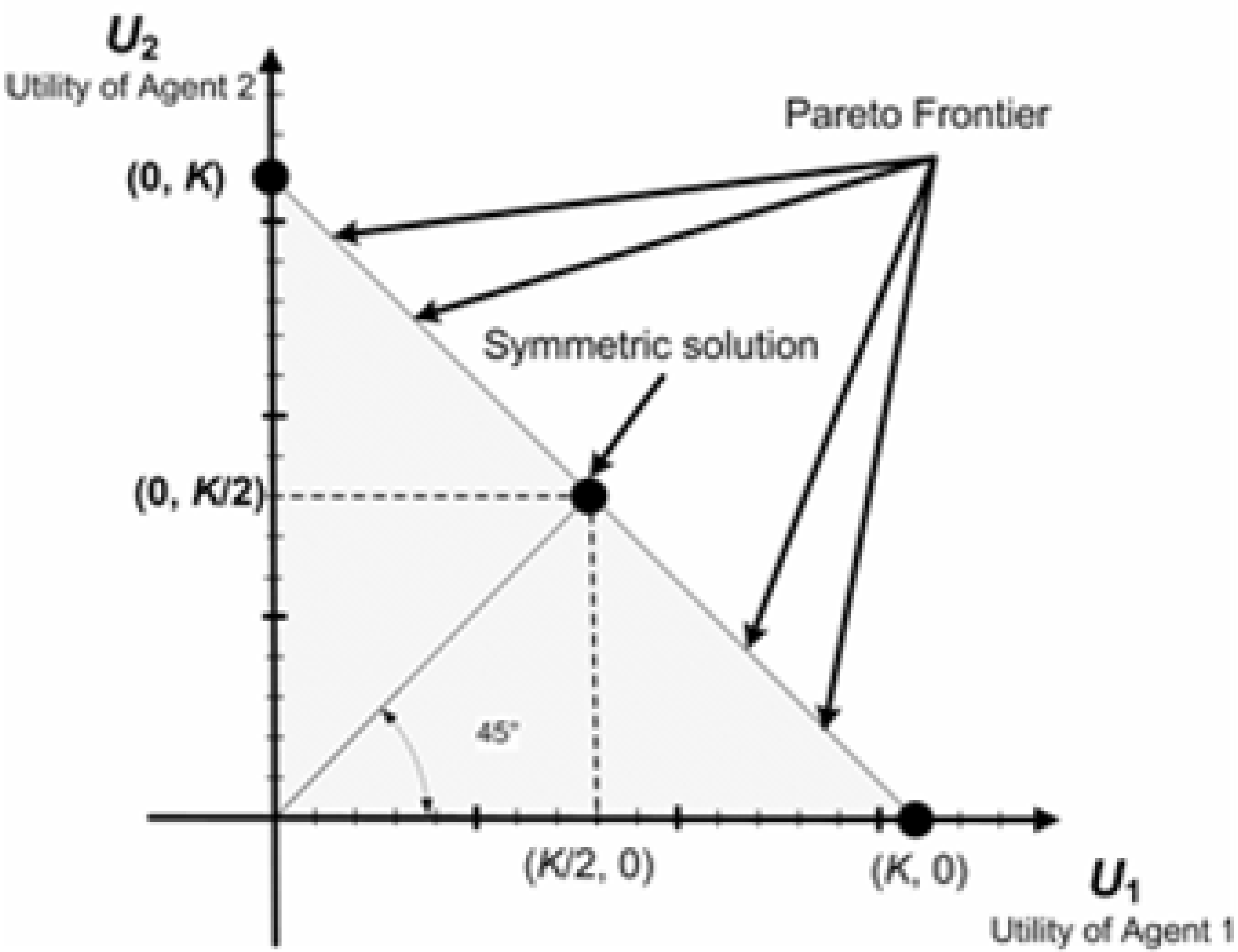
Axiom 2. *Individual rationality*



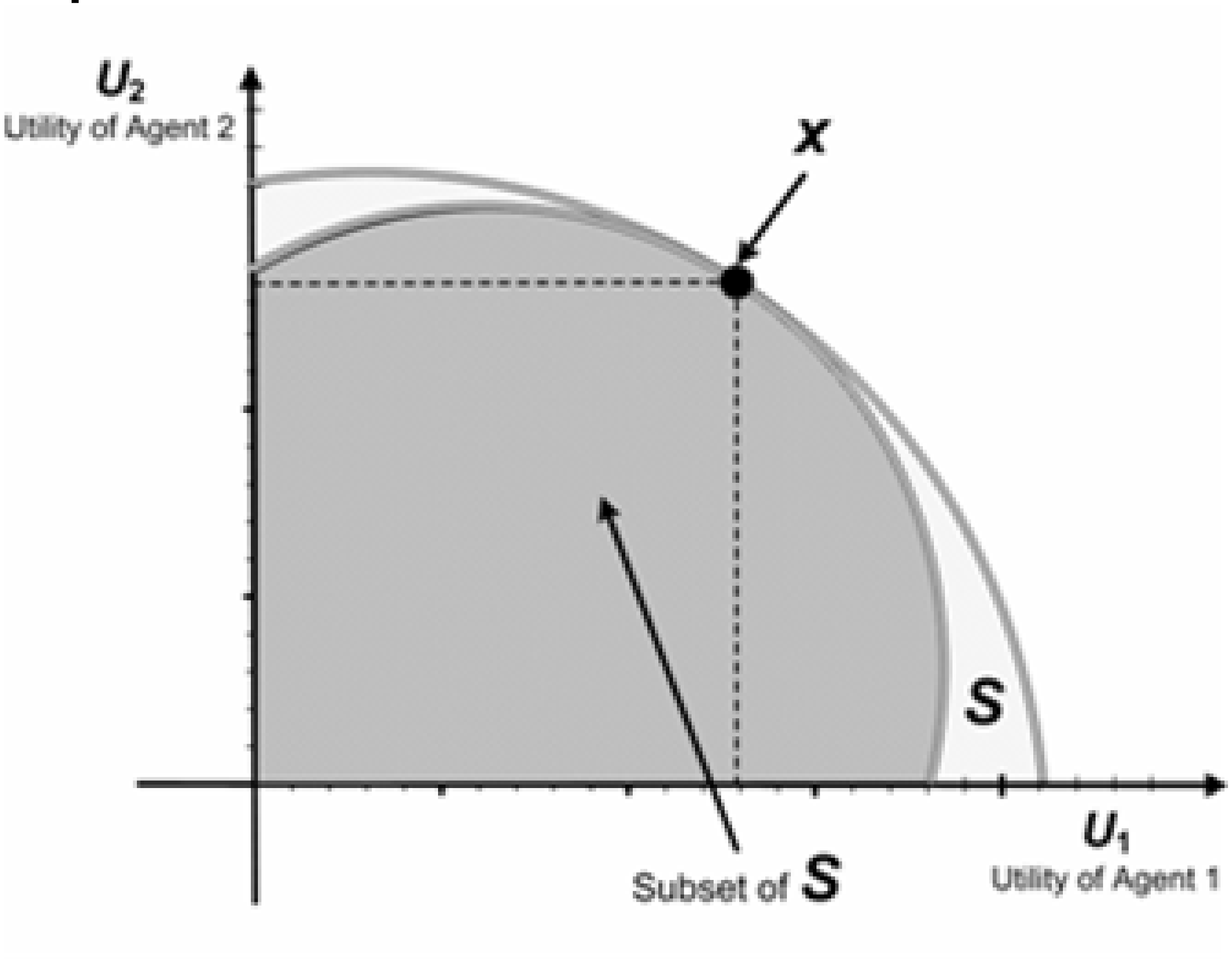
Axiom 3. *Pareto Optimality*



Axiom 4. *Symmetry*

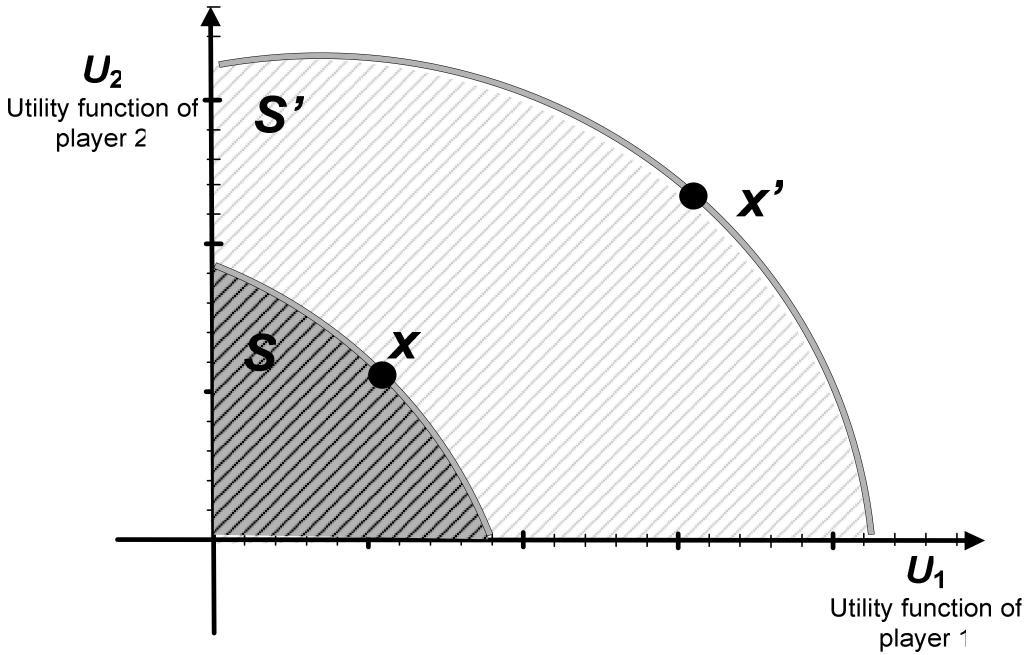


Axiom 5. *Independent of Irrelevant Alternatives*



Axiom 6. *Scale Invariance*

This implies that the solution *x* is independent of the scale that is used in measuring the players′ utilities, i.e. if we multiply the utilities of all players *i* = 1,2,3,...,*n* by constants (a_1, a_2, a_3, ... , a_n), then we have the feasible set *S′* and we get the relative solution *x′* through the multiplication of the players′ coordinates by these constants *(Jain and Mahdian, 2007).*



The Nash bargaining solution (NBS):
Nash J F, (1950). The bargaining problem, *Econometrica* 18, pp. 155–162, available at:

http://www.math.mcgill.ca/vetta/CS764.dir/nashbarg.pdf

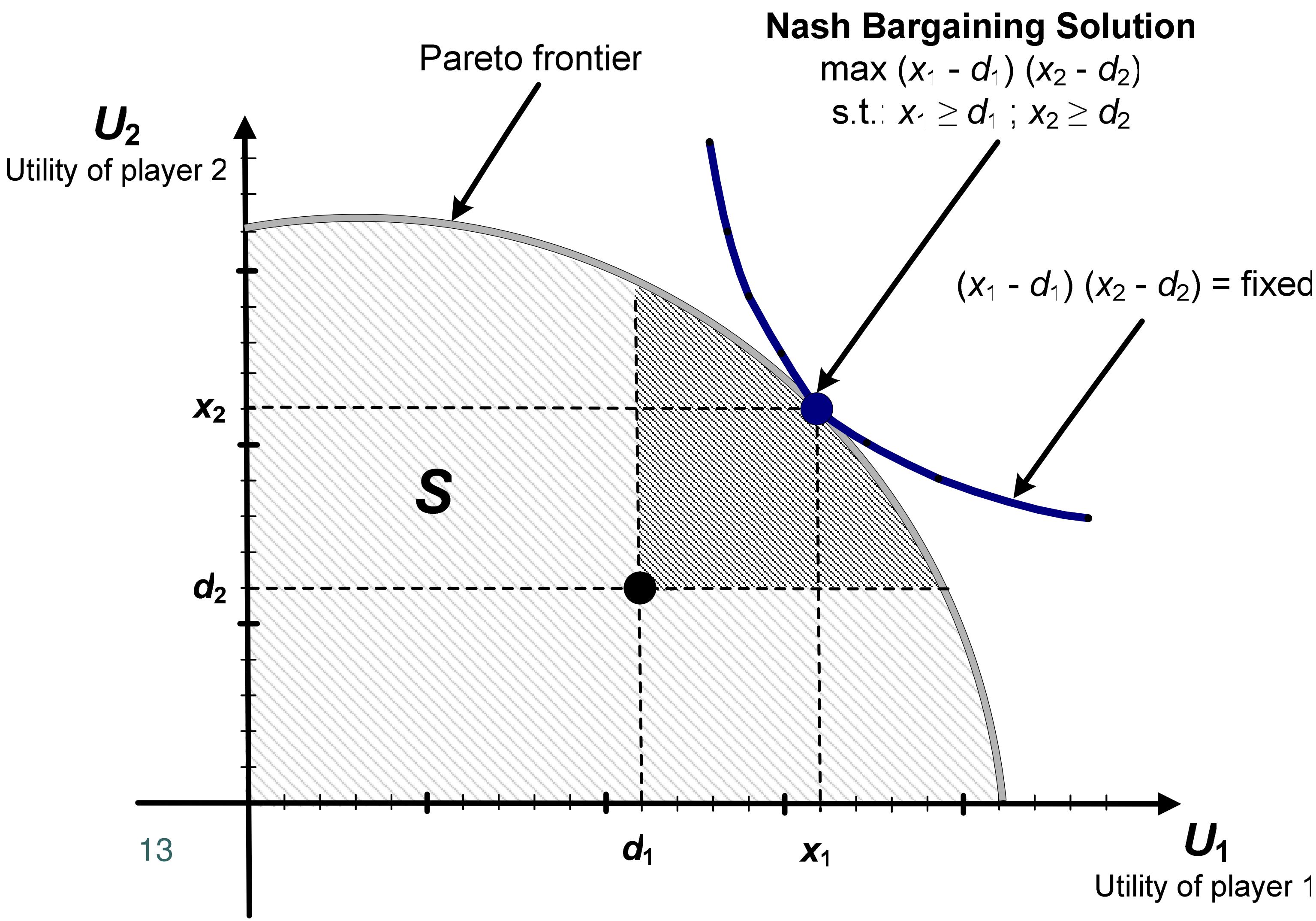
- There is a unique solution in satisfying Axioms 1 to 6, i.e. feasibility, individual rationality, Pareto optimality, symmetry, independence of irrelevant alternatives and independence of equivalent utility representations.

- Roth, A.E., (1979): *Axiomatic models of bargaining*, presents is detail all axioms used by Nash, available at:

http://kuznets.fas.harvard.edu/~aroth/Axiomatic_Models_of_Bargaining.pdf

- The NBS is a function that maximizes the geometric average of the players' gains through the agreement, instead of settling for the disagreement point d.

- In other words, the NBS for a two-person bargaining game, is the solution maximizing the product of the excesses: $(x_1 - d_1)(x_2 - d_2)$, subject to constraints: $x_1 \geq d_1$, and $x_2 \geq d_2$.



Harsanyi (1959), (1963) showed that the NBS can be easily expanded in a finite set of $n > 2$ players: $N = \{1,2, .. , n\}$:

$$\text{Max. } \prod (x_i - d_i) \,, \quad \text{s.t. } x_i \geq d_i \quad i \in N$$

- The solution of the game is a vector $x \in \mathfrak{R}^N$ representing the allocation of the overall profit $p(N)$ to each agent.

- The finite set N is called the grand-coalition, every subset in which this set can be divided is called coalition and a coalition with just one player is called singleton

In cases with identical players (equal disagreement outcomes and symmetric utility functions), then the solution is the symmetric NBS and the surplus is divided equally: $x_i = x_k, \forall\, i, k \in N$

- Kalai (1977) introduced the asymmetric NBS, as the solution maximizing the product of the excesses:

$$(x_1 - d_1)^{p} (x_2 - d_2)^{q}$$

where p, q are positive numbers such that: $p + q = 1$

Available at:

http://www.datalaundering.com/download/IntJGT-6-3-129-133.pdf

- The symmetric NBS is the special case where: $p = q = 1/2$

- That is, in cases with non-identical players (unequal disagreement outcomes or/and and asymmetric utility functions), then the surplus is divided unequally for at least two players $i, k \in N$:

$$x_i \neq x_k$$

Criticisms in the NBS about its real-life applications arise from the assumptions used in the model (in real-life these assumptions can be rarely found):

- There is full information among players regarding their utility function and their disagreement outcomes.
- Players are rational seeking to maximize their utility
- The disagreement outcomes are fixed
- The pie over which players negotiate is divisible and its size is fixed (deterministic model)

A Query:

- What happens if we relax one assumption that is the cooperation's overall return (including the surplus) consists of multiple pies with uncertain sizes.
(stochastic model)

Assumptions

A finite number of agents indexed by *i*. Let $N = \{1,2,3,\ldots,n\}$ denote the grand-coalition.

- **Assumption 1**. There is complete information among agents, who examine to cooperate having fixed disagreement payoffs.
  Payoff vector: $(C_1, C_2, \ldots, C_n)$.
  Objectives: partially cooperative – conflicting.

- **Assumption 2**. The different gains and losses that are yielded through cooperation form a finite set of pies $J = \{1,2,3,\ldots,m\}$ that is called pie-set. These pies are divisible, and are assumed to be stochastic variables. Specifically, all pies indexed by *j* follow normal probability distribution functions:

$$\Pi^j(\mu^j, \sigma^{j^2}) = \left(1/\sqrt{2\pi}\,\sigma^j\right) e^{-\frac{(P^j - \mu^j)^2}{2\sigma^{j^2}}}$$

where $\sigma^j > 0$ for all pies, $\mu^j > 0$ for gains and $\mu^j < 0$ for losses.

- **Assumption 3**. Agents are rational, i.e. each agent should get at least as much as it could obtain through the non-cooperative option. Clearly, the cooperation yields a nonnegative surplus *S*.

That is, the *N*'s overall return: $\sum_{j=1}^{m} \Pi^{j}$

equals the sum of disagreement payoffs: $\sum_{i=1}^{n} \mathrm{C}_{i}$

plus the surplus *S*: $\sum_{j=1}^{m} \Pi^{j} = \sum_{i=1}^{n} \mathrm{C}_{i} + S \Leftrightarrow S = \sum_{j=1}^{m} \Pi^{j} - \sum_{i=1}^{n} \mathrm{C}_{i}$

- **Assumption 4**. All agents are risk-neutrals, i.e. they are indifferent between the *m* pies, since they consider only the overall expected return when making investment decisions. In particular, the bargaining outcome is the NBS, which is a vector:

$$\mathrm{U} = \left(\mathrm{U}_{1}, \mathrm{U}_{2}, \ldots, \mathrm{U}_{n}\right)$$

representing the expected (or mean) individual surplus shares (dividends) which are allocated to players.

Axioms

Axiom 1. *Coalitional rationality*:

$$\mu_N = \sum_{j=1}^{m} \mu^j - \sum_{i=1}^{n} C_i > 0 \tag{1}$$

Since all agents are risk-neutrals, they negotiate over the *S*'s mean value considering their expected dividends (μ_i). The bargaining outcome is an allocation, i.e. a vector U Є $\mathfrak{R}^N$ representing the ratio of the μ_N that is divided in agent *i* Є *N*:

$$\mu_i = \mu_N (U_i) \tag{2}$$

- **Axiom 2**. *Individually rational*:

$$U_i > 0 \tag{3}$$

- **Axioms 3 and 4**. *Linear Invariance and Independence of Irrelevant Alternatives*:

  Let *F* be the feasible set of allocations. If *F'* is obtained from *F* by multiplying all agents' utilities by α_i then the solution of the new game is obtained by multiplying each agent's coordinate in the first solution by α_i. Moreover, the solution remains the same for each subset of *F* that includes the specific solution.

- **Axioms 5 and 6**. *Feasibility and Pareto-Optimality*: Efficient allocation of the overall surplus:

$$\sum_{i=1}^{n} \mu_i = \mu_N \Leftrightarrow \sum_{i=1}^{n} U_i = 1 \qquad (4)$$

- Let p_i^j denote the ratio of pie *j* Є *J* allocated to *i* Є *N.*

- Player's *i* expected dividend: $\mu_i = \sum_{j=1}^{m} \left(\mu^j p_i^j\right) - C_i$

- Efficient allocation of all pies *j* Є *J*:

$$\sum_{i=1}^{n} p_i^j = 1 \qquad (5)$$

- **Axiom 7**. *Symmetry (or asymmetry)*: All players identical, then:

$$U_1 = U_2 = U_3 = \ldots = U_n \overset{(2)}{\Leftrightarrow} \mu_1 = \mu_2 = \mu_3 = \ldots = \mu_n \qquad (6)$$

If players non-identical, then asymmetric NBS, i.e. the surplus can be divided either equally (Eq. (6)), or unequally for at least two agents *i*, *k* Є *N*.

$$x_i \neq x_k$$

Π_i: the individual stochastic surplus share allocated to *i*.

$$\Pi_i = \sum_{j=1}^{m} \left(\Pi^j p_i^j \right) - C_i \qquad (7)$$

- A solution is a $[P]_{n \times m}$ matrix, in which the 1,2,..,*n* rows denote the players and the 1,2,..,*m* columns denote the pies, i.e. each element represents the ratio of pie *j* which is allocated to agent *i*:

$$[P]_{n\times m} = \left[p_i^j\right]_{n\times m} = \left[\begin{array}{cccc} p_1^1 & p_1^2 & \ldots & p_1^m \\ p_2^1 & p_2^2 & \ldots. & p_2^m \\ \ldots & \ldots & \ldots & \ldots \\ p_n^1 & p_n^2 & \ldots. & p_n^m \end{array} \right] \qquad (8)$$

- The characteristic function:

$$[P]_{n\times m} \times \left[\Pi^j\right]_{m\times 1} - \left[C_i\right]_{n\times 1} = \left[\Pi_i\right]_{n\times 1} \qquad (9)$$

Axiom 8. *Fair (proportional) division*:

Aristotle in "Nicomachean Ethics":

"*Equals should be treated equally and unequals unequally, in proportion to the relevant inequality*"
"*What is just, is proportional*"

Following this maxim, the surplus is divided with fairness, when the stochastic dividends are distributed in proportion to the NBS, i.e.:

- Expected (mean) values: $\mu_i = \sum_{j=1}^{m} \left(\mu^j p_i^j\right) - C_i = \mu_N(U_i)$ (10)
- Standard deviations: $\frac{\mu_1}{\sigma_1} = \frac{\mu_2}{\sigma_2} = \frac{\mu_3}{\sigma_3} = \ldots = \frac{\mu_n}{\sigma_n}$ (11)

That is: $\frac{\mu_i}{\mu_k} = \frac{\sigma_i}{\sigma_k} = \frac{U_i}{U_k}, \quad \forall\, i, k \in N$ (12)

- **This implies that in the 3 Confidence Intervals:**
- 1st: $\text{probability}: \ (\mu_i - \sigma_i \le \Pi_i \le \mu_i + \sigma_i) \approx 0.6827$
- 2nd: $\text{probability}: \ (\mu_i - 2\sigma_i \le \Pi_i \le \mu_i + 2\sigma_i) \approx 0.9545$
- 3rd: $\text{probability}: \ (\mu_i - 3\sigma_i \le \Pi_i \le \mu_i + 3\sigma_i) \approx 0.9973$

For example:

If: $U = (U_1, U_2, U_3) = (0.3, 0.5, 0.2), \mu_N = 100,$

Then the sharing mechanism should satisfy: $\mu_1 = 30, \mu_2 = 50, \mu_3 = 20,$

and: $\sigma_1 = 3, \sigma_2 = 5, \sigma_3 = 2,$ or: $\sigma_1 = 0.6, \sigma_2 = 1.0, \sigma_3 = 0.4,$

where σ_i are unknowns (depend on the normal probability distributions of the pies).

In other words: $\frac{\mu_i}{\mu_k} = \frac{\sigma_i}{\sigma_k} = \frac{U_i}{U_k}, \quad \forall\, i, k = 1,2,3$

General Method.

- **Partition the pie-set $J = \{1, 2, \dots, m\}$ into two nonempty subsets, J_A and J_B, i.e.**

$$J = J_A \cup J_B \,;\; J_A \cap J_B = \emptyset$$

- **Partition the grand-coalition $N = \{1, 2, \dots, n\}$ into two nonempty coalitions, N_A and N_B, i.e.**

$$N = N_A \cup N_B \,;\; N_A \cap N_B = \emptyset$$

Stage 1

↓

The partitioning of all coalitions of agents is continued for n-1 times, until eventually all agents form singletons:

$$\{1\}, \{2\}, \{3\}, \dots, \{n\}$$

Stage 2

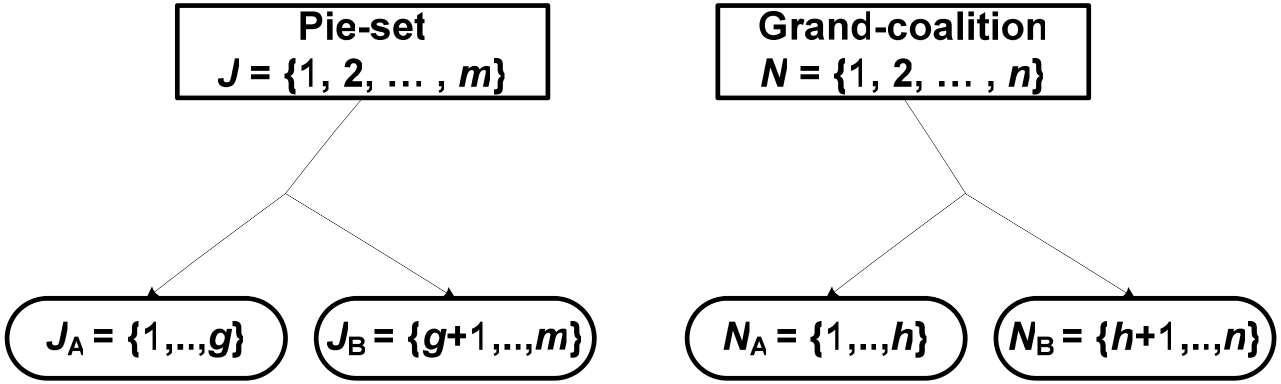
Stage 1. Partition the pie-set J into two nonempty subsets: J_A, J_B, and the grand-coalition N into two nonempty coalitions: N_A N_B.

Partition of the pie-set $J = \{1,2,..,m\}$ into two subsets: $J_A = \{1,\dots,g\}$, and $J_B = \{g+1,\dots,m\}$, with $1 \leq g < m$, the characteristic function is presented in Eq (13):

$$\begin{bmatrix} p_i^{\{1,..,g\}} & p_i^{\{g+1,..,m\}} \end{bmatrix}_{n\times 2} \begin{bmatrix} \sum_{j=1}^{g} \Pi^j \\ \sum_{j=g+1}^{m} \Pi^j \end{bmatrix}_{2\times 1} - [C_i]_{n\times 1} = [\Pi_i]_{n\times 1} \qquad (13)$$

Partition of the grand-coalition N into two coalitions: $N_A = \{1,\dots, h\}$, $N_B = \{h+1,\dots,n\}$, with $1 \leq h < n$. We have 4 unknowns, which are the ratios of each subset: $J_A = \{1,\dots,g\}$ and $J_B = \{g+1,\dots,m\}$ allocated to each coalition: $N_A = \{1,\dots, h\}$ and $N_B = \{h+1,\dots,n\}$:

$$[P]_{2\times 2} = \begin{bmatrix} p_{\{1,..,h\}}^{\{1,..,g\}} & p_{\{1,..,h\}}^{\{g+1,..m\}} \\ p_{\{h+1,..n\}}^{\{1,..,g\}} & p_{\{h+1,..n\}}^{\{g+1,..m\}} \end{bmatrix}_{2\times 2} \qquad (14)$$



Characteristic function:

$$[\mathrm{P}]_{n\times m} \times \left[\Pi^{j}\right]_{m\times 1} - \lfloor \mathrm{C}_{i} \rfloor_{n\times 1} = \lfloor \Pi_{i} \rfloor_{n\times 1} \qquad (9)$$

$$\begin{bmatrix} \mathrm{p}_{\{1,..,h\}}^{\{1,..,g\}} & \mathrm{p}_{\{1,..,h\}}^{\{g+1,..m\}} \\ \mathrm{p}_{\{h+1,..n\}}^{\{1,..,g\}} & \mathrm{p}_{\{h+1,..n\}}^{\{g+1,..m\}} \end{bmatrix}_{2\times 2} \begin{bmatrix} \sum_{j=1}^{g} \Pi^{j} \\ \sum_{j=g+1}^{m} \Pi^{j} \end{bmatrix}_{2\times 1} - \begin{bmatrix} \sum_{i=1}^{h} \mathrm{C}_{i} \\ \sum_{i=h+1}^{n} \mathrm{C}_{i} \end{bmatrix}_{2\times 1} = \begin{bmatrix} \Pi_{\{1,..,h\}} \\ \Pi_{\{h+1,..,n\}} \end{bmatrix}_{2\times 1} \qquad (15)$$

Theorem 1. *For each pair of two coalitions and two subsets that can arise from the partition of the grand-coalition* $N = \{1,..,n\}$ into:

$N_A = \{1,..,h\}$ and $N_B = \{h+1,..,n\}$, $(N_A \cup N_B = N;\ N_A \cap N_B = \emptyset)$

and the partition of the pie-set $J = \{1,..,m\}$ into:

$J_A = \{1,..,g\}$ and $J_B = \{g+1,..,m\}$, $(J_A \cup J_B = J;\ J_A \cap J_B = \emptyset)$,

there is a unique $[P]_{2 \times 2}$ *matrix:*

$$\begin{bmatrix} p_{\{1,..,h\}}^{\{1,..,g\}} & p_{\{1,..,h\}}^{\{g+1,..m\}} \\ p_{\{h+1,..n\}}^{\{1,..,g\}} & p_{\{h+1,..n\}}^{\{g+1,..m\}} \end{bmatrix}_{2\times 2}$$

, *which ensures fairness within the NBS satisfying:*

$$\frac{\mu_{\{1,..,h\}}}{\mu_{\{h+1,..,n\}}} = \frac{\sigma_{\{1,..,h\}}}{\sigma_{\{h+1,..,n\}}} = \frac{\sum_{i=1}^{h} U_i}{\sum_{i=h+1}^{n} U_i}$$

Stage 2. Continuous partitions of the coalitions for n - 1 times

Partition of $N_A = \{1,..,h\}$ into coalitions $\{1,..., f\}$,$\{f+1,\dots,h\}$, $1 \le f < h$:

$$\begin{bmatrix} p^{\{1,..,g\}}_{\{1,..,f\}} & p^{\{g+1,..m\}}_{\{1,..,f\}} \\ p^{\{1,..,g\}}_{\{f+1,..h\}} & p^{\{g+1,..m\}}_{\{f+1,..h\}} \end{bmatrix}_{2\times2} \begin{bmatrix} p^{\{1,..,g\}}_{\{1,..,h\}} \sum_{j=1}^{g} \Pi^j \\ p^{\{g+1,..m\}}_{\{1,..,h\}} \sum_{j=g+1}^{m} \Pi^j \end{bmatrix}_{2\times1} - \begin{bmatrix} \sum_{i=1}^{f} C_i \\ \sum_{i=f+1}^{h} C_i \end{bmatrix}_{2\times1} = \begin{bmatrix} \Pi_{\{1,..,f\}} \\ \Pi_{\{f+1,..h\}} \end{bmatrix}_{2\times1} \quad (16)$$

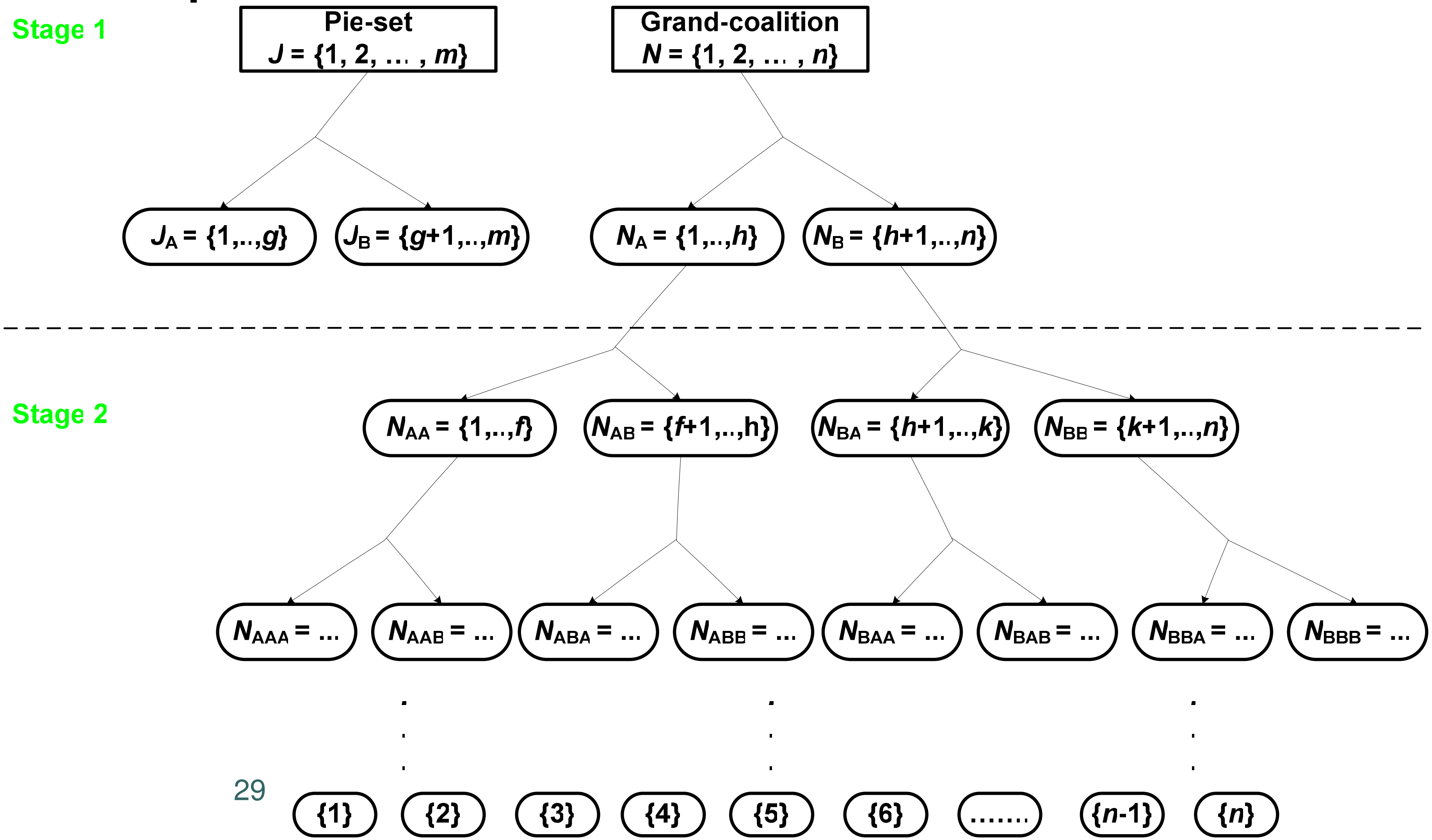
Which has a unique matrix: $\begin{bmatrix} p^{\{1,..,g\}}_{\{1,..,f\}} & p^{\{g+1,..m\}}_{\{1,..,f\}} \\ p^{\{1,..,g\}}_{\{f+1,..h\}} & p^{\{g+1,..m\}}_{\{f+1,..h\}} \end{bmatrix}_{2\times2}$

Partition of $N_B = \{h+1,\dots,n\}$ into $\{h+1,\dots,k\}$,$\{k+1,\dots,n\}$, $h+1 \le k < n$:

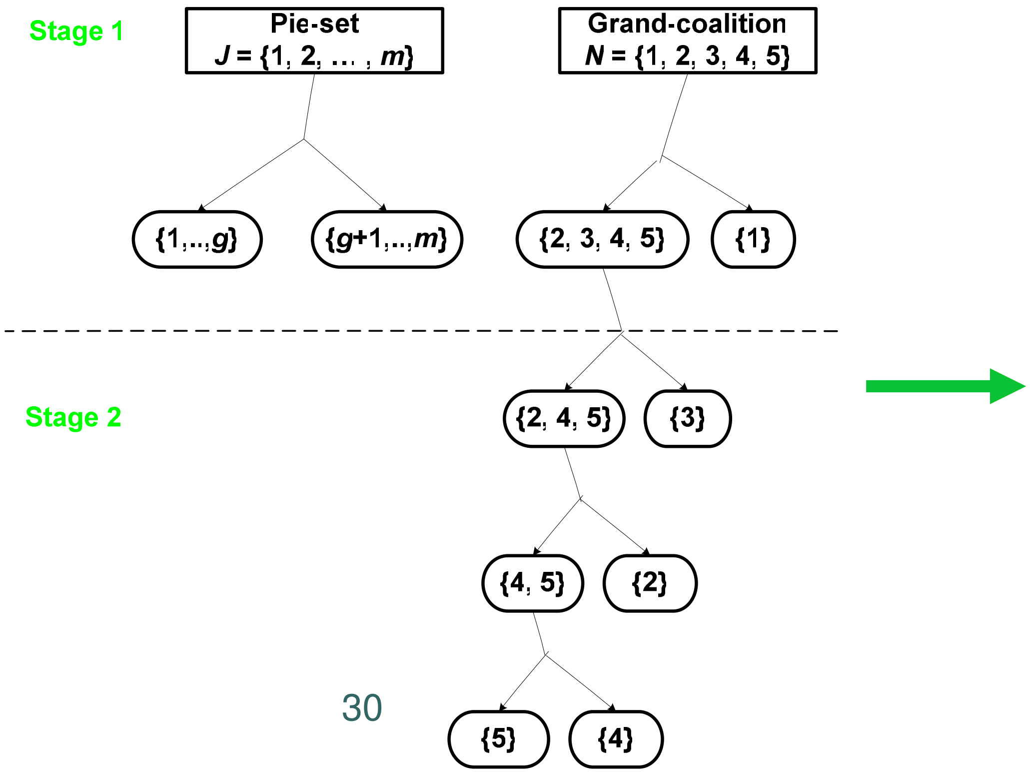
$$\begin{bmatrix} p^{\{1,..,g\}}_{\{h+1,..,k\}} & p^{\{g+1,..m\}}_{\{h+1,..,k\}} \\ p^{\{1,..,g\}}_{\{k+1,..n\}} & p^{\{g+1,..m\}}_{\{k+1,..n\}} \end{bmatrix}_{2\times2} \begin{bmatrix} p^{\{1,..,g\}}_{\{h+1,..,n\}} \sum_{j=1}^{g} \Pi^j \\ p^{\{g+1,..,m\}}_{\{h+1,..,n\}} \sum_{j=g+1}^{m} \Pi^j \end{bmatrix}_{2\times1} - \begin{bmatrix} \sum_{i=h+1}^{k} C_i \\ \sum_{i=k+1}^{n} C_i \end{bmatrix}_{2\times1} = \begin{bmatrix} \Pi_{\{h+1,..,k\}} \\ \Pi_{\{k+1,..,n\}} \end{bmatrix}_{2\times1} \quad (17)$$

Which also has a unique matrix: $\begin{bmatrix} p^{\{1,..,g\}}_{\{h+1,..,k\}} & p^{\{g+1,..m\}}_{\{h+1,..,k\}} \\ p^{\{1,..,g\}}_{\{k+1,..n\}} & p^{\{g+1,..m\}}_{\{k+1,..n\}} \end{bmatrix}_{2\times2}$

In particular, the partitioning of coalitions into two nonempty coalitions is continued, until eventually all agents form singletons: {1}, {2},…, {n}, i.e. for n-1 times.



- Through this process, if we compute the unique $[P]_{2 \times 2}$ matrix for each coalition, we can compute the ratio of each subset $\{1,..,g\}$ and $\{g+1,..,m\}$ which is allocated to each agent.
- Let π_i denote the set of coalitions including agent i, which arise from a specific set of partitions of the grand-coalition N into two nonempty coalitions for n-1 times.
- For instance: Partitions of the $N = \{1,2,3,4,5\}$ into: {1} and {2,3,4,5}, and further into {3} and {2,4,5}, and further into {2} and {4,5}, and further into {4} and {5}. The sets of coalitions including agents are:



$$\pi_1 = \{\{1\}\}$$
$$\pi_2 = \{\{2,3,4,5\},\{2,4,5\},\{2\}\}$$
$$\pi_3 = \{\{2,3,4,5\},\{3\}\}$$
$$\pi_4 = \{\{2,3,4,5\},\{2,4,5\},\{4,5\},\{4\}\}$$
$$\pi_5 = \{\{2,3,4,5\},\{2,4,5\},\{4,5\},\{5\}\}$$

- That is, the ratio of each subset of pies ({1,..,*g*} and {*g*+1,..,*m*}), which is allocated to agent *i*, equals the product of ratios of the coalitions, in which *i* is included, e.g. for agent 5:

$$p_5^{\{1,..,g\}} = \prod_{k \in \pi_5} p_k^{\{1,..,g\}} = (p_{\{2,3,4,5\}}^{\{1,..,g\}})(p_{\{2,4,5\}}^{\{1,..,g\}})(p_{\{4,5\}}^{\{1,..,g\}})(p_{\{5\}}^{\{1,..,g\}})$$

$$p_5^{\{g+1,..,m\}} = \prod_{k \in \pi_5} p_k^{\{g+1,..,m\}} = (p_{\{2,3,4,5\}}^{\{g+1,..,m\}})(p_{\{2,4,5\}}^{\{g+1,..,m\}})(p_{\{4,5\}}^{\{g+1,..,m\}})(p_{\{5\}}^{\{g+1,..,m\}})$$

- However, the ratio for agent *i* in each subset of pies {1,…,*g*} and {*g*+1,…,*m*}, equals her ratios in all pies of the specific subset:

$$p_5^{\{1,..,g\}} = p_5^1 = p_5^2 = \ldots = p_5^g$$

$$p_5^{\{g+1,..,m\}} = p_5^{g+1} = p_5^{g+2} = \ldots = p_5^m$$

- In other words, we compute the ratios of all pies $j = 1,2,..,m$ which are allocated to all agents $i = 1,2,..,n$. That is, we can compute a specific matrix $[P]_{n \times m}$, which ensures that the agents' dividends are distributed in proportion to the NBS:

$$\frac{\mu_i}{\mu_k} = \frac{\sigma_i}{\sigma_k} = \frac{U_i}{U_k}, \quad \forall\, i, k \in N$$

Number of possible partitions of the pie-set

Proposition 1. *The number of possible partitions of a pie-set J into two nonempty subsets is given from the piecewise Eq (18)*:

$$g(m) = \frac{m!}{(m-1)!} + \sum_{l=2}^{\frac{m-1}{2}} \frac{m!}{(m-l)!\,l!}, \quad m = \text{περιτος}$$

$$g(m) = \frac{m!}{(m-1)!} + \sum_{l=2}^{\frac{m}{2}-1} \frac{m!}{(m-l)!\,l!} + \frac{m!}{(\frac{m}{2}!)^2}\frac{1}{2}, \quad m = \text{αρτιος} \qquad (18)$$

Finite possible [P] $_{n \times m}$ matrices for fair surplus division

Theorem 2. *The number of possible* [P] $_{n \times m}$ *matrices that ensure fairness for the surplus division within a NBS, is finite and equals the product of the possible partitions of the pie-set into two subsets with the possible partitions of all coalitions into two nonempty coalitions for n-1 times:*

$$\text{possible } [\mathrm{P}]_{n \times m} = f(n)\,g(m) \qquad (19)$$

$$f(n)=\frac{n!}{(n-1)!}f(n-1)+\sum_{k=2}^{\frac{n-1}{2}}\frac{n!}{(n-k)!k!}f(n-k)f(k),\quad n=\pi\varepsilon\rho\iota\tau o\varsigma$$

$$f(n)=\frac{n!}{(n-1)!}f(n-1)+\sum_{k=2}^{\frac{n}{2}-1}\frac{n!}{(n-k)!k!}f(n-k)f(k)+\frac{n!}{(\frac{n}{2}!)^2}f(\frac{n}{2}!)^2\frac{1}{2},\quad n=\alpha\rho\tau\iota o\varsigma \qquad (19.1)$$

$$g(m)=\frac{m!}{(m-1)!}+\sum_{l=2}^{\frac{m-1}{2}}\frac{m!}{(m-l)!l!},\quad m=\pi\varepsilon\rho\iota\tau o\varsigma$$

$$g(m)=\frac{m!}{(m-1)!}+\sum_{l=2}^{\frac{m}{2}-1}\frac{m!}{(m-l)!l!}+\frac{m!}{(\frac{m}{2}!)^2}\frac{1}{2},\quad m=\alpha\rho\tau\iota o\varsigma \qquad (19.2)$$

Computation of possible [P] $_{n\,X\,m}$ matrices for fair surplus division with the Wolfram Research Mathematica:

$$\texttt{f}[n\_] := \texttt{Piecewise}\Big[\Big\{\Big\{\frac{n\,!}{(n-1)\,!}\,\texttt{f}[n-1] + \sum_{k=2}^{\frac{n-1}{2}}\left(\frac{n\,!}{(n-k)\,!\,k\,!}\,\texttt{f}[n-k]\,\texttt{f}[k]\right),\ \frac{n-1}{2} \in \texttt{Integers}\ \&\&\ n \geq 3\Big\},$$

$$\Big\{\frac{n\,!}{(n-1)\,!}\,\texttt{f}[n-1] + \sum_{k=2}^{\frac{n}{2}-1}\left(\frac{n\,!}{(n-k)\,!\,k\,!}\,\texttt{f}[n-k]\,\texttt{f}[k]\right) + \frac{n\,!}{\left(\frac{n}{2}\,!\right)^2}\,\texttt{f}\left[\frac{n}{2}\right]^2\,\frac{1}{2},\ \frac{n}{2} \in \texttt{Integers}\ \&\&\ n \geq 4\Big\}\Big\},\ 1\Big]$$

$$\texttt{g}[m\_] := \texttt{Piecewise}\Big[\Big\{\Big\{\frac{m\,!}{(m-1)\,!} + \sum_{l=2}^{\frac{m-1}{2}}\left(\frac{m\,!}{(m-l)\,!\,l\,!}\right),\ \frac{m-1}{2} \in \texttt{Integers}\ \&\&\ m \geq 3\Big\},$$

$$\Big\{\frac{m\,!}{(m-1)\,!} + \sum_{l=2}^{\frac{m}{2}-1}\left(\frac{m\,!}{(m-l)\,!\,l\,!}\right) + \frac{m\,!}{\left(\frac{m}{2}\,!\right)^2}\,\frac{1}{2},\ \frac{m}{2} \in \texttt{Integers}\ \&\&\ m \geq 4\Big\}\Big\},\ 1\Big]$$

```
MatrixForm[Table[Table[f[n] g[m], {m, 2, 10}], {n, 2, 10}]]
```

Number of possible [P] $_{n \times m}$ matrices for fair surplus division

		Number of pies								
		$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
Number of Agents	$n=2$	1	3	7	15	31	63	127	255	511
	$n=3$	3	9	21	45	93	189	381	765	1533
	$n=4$	15	45	105	225	465	945	1905	3825	7665
	$n=5$	105	315	735	1575	3255	6615	13335	26775	53655
	$n=6$	945	2835	6615	14175	29295	59535	120015	240975	482895
	$n=7$	10395	31185	72765	155925	322245	654885	1320165	2650725	5311845
	$n=8$	135135	405405	945945	2027025	4189185	8513505	17162145	34459425	69053985
	$n=9$	2027025	6081075	14189175	30405375	62837775	127702575	257432175	516891375	1035809775
	$n=10$	34429425	103378275	241215975	516891375	1068242175	2170943775	4376346975	8787153375	17608766175

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- **Step 1.** Randomly partition the pie-set J into two subsets: $\{1,\ldots,g\}$ and $\{g+1,\ldots,m\}$ with $1 \le g < m$.
- **Step 2.** Randomly partition the grand-coalition N into two coalitions: $\{1,\ldots,h\}$ and $\{h+1,\ldots,n\}$ with $1 \le h < n$.
- **Step 3.** Develop a Monte Carlo simulation model, in which the C_i , Π^j are inputs and the $\Pi_{\{1,..,h\}}$, $\Pi_{\{h+1,..,n\}}$ the outputs, according to:

$$\Pi_{\{1,..,h\}} = p_{\{1,..,h\}}^{\{1,..,g\}} \sum_{j=1}^{g} \Pi^j + p_{\{1,..,h\}}^{\{g+1,..m\}} \sum_{j=g+1}^{m} \Pi^j - \sum_{i=1}^{h} C_i \qquad (20)$$

$$\Pi_{\{h+1,..n\}} = p_{\{h+1,..n\}}^{\{1,..,g\}} \sum_{j=1}^{g} \Pi^j + p_{\{h+1,..n\}}^{\{g+1,..m\}} \sum_{j=g+1}^{m} \Pi^j - \sum_{i=h+1}^{n} C_i \qquad (21)$$

- **Step 4.** Select a specific value of $p_{\{1,..,h\}}^{\{1,..,g\}} = 1 - p_{\{h+1,..,n\}}^{\{1,..,g\}}$ and estimate the $p_{\{1,..,h\}}^{\{g+1,..,m\}} = 1 - p_{\{h+1,..,n\}}^{\{g+1,..,m\}}$ in order to fulfill:

$$\mu_{\{1,..,h\}} = \mu_N \sum_{i=1}^{h} U_i = p_{\{1,..,h\}}^{\{1,..,g\}} \sum_{j=1}^{g} \mu^j + p_{\{1,..,h\}}^{\{g+1,..m\}} \sum_{j=g+1}^{m} \mu^j - \sum_{i=1}^{h} C_i \qquad (22)$$

$$\mu_{\{h+1,..n\}} = \mu_N \sum_{i=h+1}^{n} U_i = p_{\{h+1,..n\}}^{\{1,..,g\}} \sum_{j=1}^{g} \mu^j + p_{\{h+1,..n\}}^{\{g+1,..m\}} \sum_{j=g+1}^{m} \mu^j - \sum_{i=h+1}^{n} C_i \qquad (23)$$

For the scenario that fulfils Eqs (22), (23) run the MCS and estimate $\sigma_{\{1,..,h\}}$, $\sigma_{\{h+1,..,n\}}$. If the following Eq (24) is fulfilled then go the next Step, otherwise examine alternative values (simply by increasing / decreasing its initial value), until you find the unique $[P]_{2\times 2}$ matrix:

$$\frac{\mu_{\{1,..,h\}}}{\mu_{\{h+1,..n\}}} = \frac{\sigma_{\{1,..,h\}}}{\sigma_{\{h+1,..n\}}} = \frac{\sum_{i=1}^{h} U_i}{\sum_{i=h+1}^{n} U_i} \qquad (24)$$

- **Step 5.** Use the $(p^{\{1,..,g\}}_{\{1,..,h\}}, p^{\{1,..,g\}}_{\{h+1,..,n\}}, p^{\{g+1,..,m\}}_{\{1,..,h\}}, p^{\{g+1,..,m\}}_{\{h+1,..,n\}})$, and return to Step 2, i.e. randomly partition both the $\{1,\ldots,h\}$ and $\{h+1,\ldots,n\}$ coalitions into two pairs of nonempty coalitions: $\{1,\ldots,f\}$, $\{f+1,\ldots,h\}$ and $\{h+1,\ldots,k\}$, $\{k+1,\ldots,n\}$, respectively, and compute the unique ratios. Specifically, the 2 to 5 Steps should be followed for n-1 times.
- **Step 6.** Compute the ratio of each subset $\{1,\ldots,g\}$ and $\{g+1,\ldots,m\}$ allocated to each agent, through the product of the ratios of the agent-coalitions in which the agent is included. Moreover, each agent's ratios are equal in all pies of each subset, so illustrate the $[P]_{n\times m}$ matrix.

- **Step 7.** Develop another MCS model, in which the computed $[P]_{n \times m}$, the $[\Pi^j]_{m \times 1}$, and the $[C_i]_{n \times 1}$ matrices are inputs, and the $[\Pi_i]_{n \times 1}$ is the output according to Eq (9):

$$[P]_{n\times m} \times [\Pi^{j}]_{m\times 1} - \lfloor C_i \rfloor_{n\times 1} = \lfloor \Pi_i \rfloor_{n\times 1} \qquad (9)$$

- Run the simulation and estimate:

$\mu_1, \mu_2, \mu_3, ..., \mu_n$, and $\sigma_1, \sigma_2, \sigma_3, ..., \sigma_n$,

in order to verify that the dividends are allocated to all agents with fairness:

$$\frac{\mu_i}{\mu_k} = \frac{\sigma_i}{\sigma_k} = \frac{U_i}{U_k}, \quad \forall\, i, k \in N$$

A consortium of scholars for a specific public project.

The project concerns a set of technical studies for a construction project on behalf of the MoD in 2011. According to the tender Call, there are 8 different studies to be developed by a consortium of studiers, each with specific qualification requirements:

No	Category / Class Degree	Type of Study	Pre-estimation of Remunerations (€)
1	-	Tender	63.927,73
2	-	Safety	8.427,99
3	6 / Β′ or Γ′	Architecture	43.980,64
4	8 / Γ′ or Δ′	Static	46.674,04
5	9 / Ε′	Electro mechanic	564.722,18
6	10 / Α′	Transportation	3.706,14
7	13 / Α′	Hydraulic	10.969,22
8	21 / Γ′	Geotechnical	86.168,52
		Total	828.576,46

Following this Call, a specific consortium of 6 scholars was formed, and after negotiation, they have agreed in the following:

Player	Class Degree	Allocation of Remunerations	
		Study (100%)	Tender + Safety **(Surplus)**
1	06 Β	Architecture	5,82%
2	08 Γ	Static	6,17%
3	09 Ε	Electro mechanic	74,68%
4	10 Α	Transportation	0,49%
5	13 Α	Hydraulic	1,45%
6	21 Γ	Geotechnical	11,39%
		Total	100,00%

Notations.

- Grand-coalition (studiers): $N = \{1,2,3,4,5,6\}$
- Pie-set (studies): $J = \{1,2,3,4,5,6,7,8\}$
- Bargaining outcome for surplus division:

$$U = (U_1, U_2, U_3, U_4, U_5, U_6) = (0.0582, 0.0617, 0.7468, 0.0049, 0.0145, 0.1139)$$

- Profit-sharing mechanism:

$$[P]_{6\times8} = [p_i^j]_{6\times8} = \begin{bmatrix} p_1^1 & p_1^2 & p_1^3 & p_1^4 & p_1^5 & p_1^6 & p_1^7 & p_1^8 \\ p_2^1 & p_2^2 & p_2^3 & p_2^4 & p_2^5 & p_2^6 & p_2^7 & p_2^8 \\ p_3^1 & p_3^2 & p_3^3 & p_3^4 & p_3^5 & p_3^6 & p_3^7 & p_3^8 \\ p_4^1 & p_4^2 & p_4^3 & p_4^4 & p_4^5 & p_4^6 & p_4^7 & p_4^8 \\ p_5^1 & p_5^2 & p_5^3 & p_5^4 & p_5^5 & p_5^6 & p_5^7 & p_5^8 \\ p_6^1 & p_6^2 & p_6^3 & p_6^4 & p_6^5 & p_6^6 & p_6^7 & p_6^8 \end{bmatrix}$$

- Such that the players' profits: $\Pi_i(\mu_i, \sigma_i^2)$ satisfy the axiom of fair (proportional) division: $\frac{\mu_i}{\mu_k} = \frac{\sigma_i}{\sigma_k} = \frac{U_i}{U_k}, \quad \forall\, i, k \in N$

- 6 out of 8 pies are indivisible, since the "Architect study (no 3) can be developed only by the Architect, the Static study (no 4) only by the Civil Engineering, etc):

$$[P]_{6\times 8} = \left[p_i^j\right]_{6\times 8} = \begin{bmatrix} p_1^1 & p_1^2 & 1 & 0 & 0 & 0 & 0 & 0 \\ p_2^1 & p_2^2 & 0 & 1 & 0 & 0 & 0 & 0 \\ p_3^1 & p_3^2 & 0 & 0 & 1 & 0 & 0 & 0 \\ p_4^1 & p_4^2 & 0 & 0 & 0 & 1 & 0 & 0 \\ p_5^1 & p_5^2 & 0 & 0 & 0 & 0 & 1 & 0 \\ p_6^1 & p_6^2 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Since there is uncertainty over the size of the remunerations that will be finally achieved by the grand-coalition *N*, a normal probability distribution function is assigned in each pie

A/A	Type of Study	Mean value: μ^j	Standard deviation: σ^j
$j = 1$	Tender	63.927,73	28.767,47
$j = 2$	Safety	8.427,99	3.792,59
$j = 3$	Architecture	43.980,64	2.199,03
$j = 4$	Static	46.674,04	2.333,70
$j = 5$	Electro mechanic	564.722,18	28.236,11
$j = 6$	Transportation	3.706,14	185,31
$j = 7$	Hydraulic	10.969,22	548,46
$j = 8$	Geotechnical	86.168,52	4.308,43

- Initially, we examine the case where each of the divisible pies (No 1: Tender, and No. 2: Safety), is allocated proportionally among agents:

$$[P]_{6\times 8} = [p_i^j]_{6\times 8} = \begin{bmatrix} 0{,}0582 & 0{,}0582 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0{,}0617 & 0{,}0617 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0{,}7468 & 0{,}7468 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0{,}0049 & 0{,}0049 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0{,}0145 & 0{,}0145 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0{,}1139 & 0{,}1139 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- That is we use the above profit-sharing mechanism as input in a Monte Carlo model within the characteristic function:

$$[P]_{6\times 8}[\Pi^j]_{8\times 1} = [\Pi_i]_{8\times 1}$$

- The simulation gives the following results:

$$\mu_1 = 48191.6, \mu_2 = 51138.1, \mu_3 = 618756.7, \mu_4 = 4060.6, \mu_5 = 12018.3, \mu_6 = 94409.5$$

$$\sigma_1 = 2803.5, \sigma_2 = 2967.3, \sigma_3 = 35365.5, \sigma_4 = 233.6, \sigma_5 = 687.5, \sigma_6 = 5452.4$$

- These results verify that this mechanism is unfair, because:

$$\frac{\mu_1}{\mu_2}=\frac{U_1}{U_2}=\frac{0.0582}{0.0617},\ \frac{\mu_1}{\mu_3}=\frac{U_1}{U_3}=\frac{0.0582}{0.7468},\ \frac{\mu_1}{\mu_4}=\frac{U_1}{U_4}=\frac{0.0582}{0.0049},\ \frac{\mu_1}{\mu_5}=\frac{U_1}{U_5}=\frac{0.0582}{0.0145},$$

$$\frac{\mu_1}{\mu_6}=\frac{U_1}{U_6}=\frac{0.0582}{0.1139},\ \frac{\mu_2}{\mu_3}=\frac{U_2}{U_3}=\frac{0.0617}{0.7468},\ \frac{\mu_2}{\mu_4}=\frac{U_2}{U_4}=\frac{0.0617}{0.0049},\ldots\ldots,\frac{\mu_5}{\mu_6}=\frac{U_5}{U_6}=\frac{0.0145}{0.1139}$$

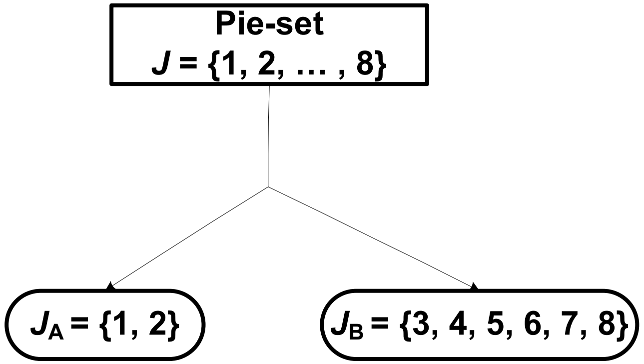
- However, the profits’ standard deviations are not in proportion to the bargaining outcome U:

$$\frac{\sigma_1}{\sigma_2}\neq\frac{\mu_1}{\mu_2}=\frac{U_1}{U_2},\ \frac{\sigma_1}{\sigma_3}\neq\frac{\mu_1}{\mu_3}=\frac{U_1}{U_3},\ \frac{\sigma_1}{\sigma_4}\neq\frac{\mu_1}{\mu_4}=\frac{U_1}{U_4},\ \frac{\sigma_1}{\sigma_5}\neq\frac{\mu_1}{\mu_5}=\frac{U_1}{U_5},\ \frac{\sigma_1}{\sigma_6}\neq\frac{\mu_1}{\mu_6}=\frac{U_1}{U_6},$$

$$\frac{\sigma_2}{\sigma_3}\neq\frac{\mu_2}{\mu_3}=\frac{U_2}{U_3},\ \frac{\sigma_2}{\sigma_4}\neq\frac{\mu_2}{\mu_4}=\frac{U_2}{U_4},\ldots\ldots\ldots,\frac{\sigma_5}{\sigma_6}\neq\frac{\mu_5}{\mu_6}=\frac{U_5}{U_6}$$

- Further, we follow the computation algorithm

- Due to the fact that some pies are indivisible, a partition of the pie-set is followed into two sets: J_A (divisible subset) and J_B (indivisible subset):



Set of divisible pies   Set of indivisible pies

- i.e. the set of indivisible pies is fixed:

$$[\mathrm{P}]_{6\times 6}^{j\in J_B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- and we look for a $[\mathrm{P}]_{6\times 2}^{j\in J_A}$ matrix:

$$[\mathrm{P}]_{6\times 2}^{j\in J_A} = \begin{bmatrix} p_1^1 & p_1^2 \\ p_2^1 & p_2^2 \\ p_3^1 & p_3^2 \\ p_4^1 & p_4^2 \\ p_5^1 & p_5^2 \\ p_6^1 & p_6^2 \end{bmatrix}$$

- Characteristic function: $[\mathrm{P}]_{6\times 2}^{j\in J_A}[\Pi^j]_{2\times 1}^{j\in J_A} + [\mathrm{P}]_{6\times 6}^{j\in J_B}[\Pi^j]_{6\times 1}^{j\in J_B} = [\Pi_i]_{6\times 1}$

- **Step 1.** Partition the pie-set J into: {1} and {2}.
- **Step 2.** Random partition of the grand-coalition N into two coalitions: {2, 3, 5, 6} and {1, 4}.
- **Step 3.** Develop a Monte Carlo simulation model, in which the $[\mathrm{P}]_{6\times 2}^{j\in J_A}, [\Pi^j]_{2\times 1}^{j\in J_A}, [\mathrm{P}]_{6\times 6}^{j\in J_B}, [\Pi^j]_{6\times 1}^{j\in J_B}$ are inputs and the $\Pi_{\{2356\}}$, $\Pi_{\{14\}}$ the outputs, according to:

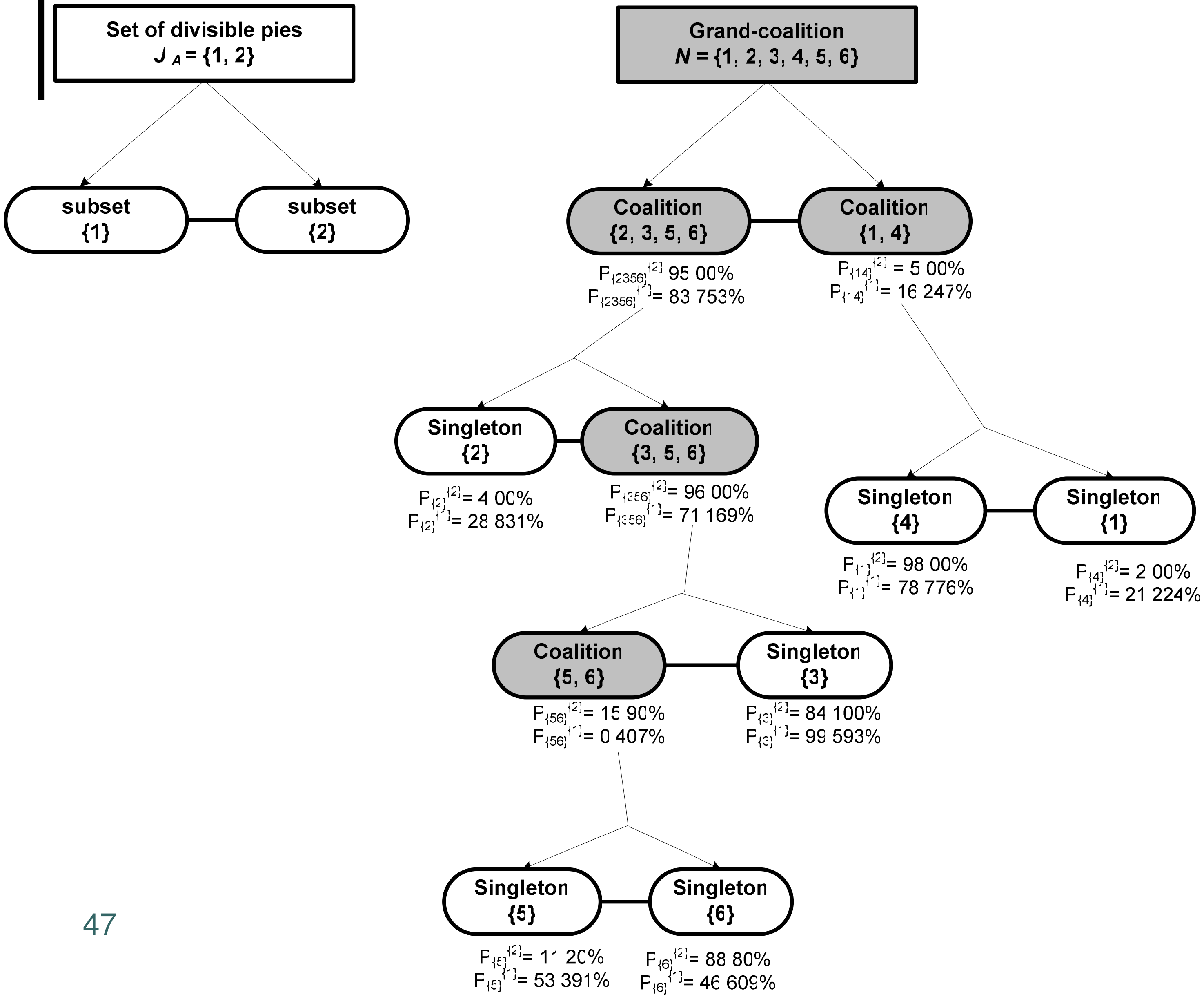
$$\Pi_{\{2356\}} = \mathrm{p}_{\{2356\}}^{\{1\}}\Pi^1 + \mathrm{p}_{\{2356\}}^{\{2\}}\Pi^2 + \Pi^4 + \Pi^5 + \Pi^7 + \Pi^8 \tag{20}$$

$$\Pi_{\{14\}} = \mathrm{p}_{\{14\}}^{\{1\}}\Pi^1 + \mathrm{p}_{\{14\}}^{\{2\}}\Pi^2 + \Pi^3 + \Pi^6 \tag{21}$$

- **Step 4.** We examine alternative values of $\mathrm{p}_{\{2356\}}^{\{2\}}$ and we compute the unique elements $\left(\mathrm{p}_{\{2356\}}^{\{2\}} = 0.95, \mathrm{p}_{\{14\}}^{\{2\}} = 0.05, \mathrm{p}_{\{2356\}}^{\{1\}} = 0.8375, \mathrm{p}_{\{14\}}^{\{1\}} = 0.1624\right)$

which fulfill: $$\frac{\mu_{\{2356\}}}{\mu_{\{14\}}} = \frac{\sigma_{\{2356\}}}{\sigma_{\{14\}}} = \frac{U_2 + U_3 + U_5 + U_6}{U_1 + U_4} = \frac{0.9369}{0.0631}$$

- **Step 5.** We return to step 2 and we follow the 2 to 5 steps for $n - 1 = 6 - 1 = 5$ times, following specific partitions:



- **Step 6.** We compute the ratios for each agent in each subset (1}, (2}:

- Agent 1: $p_1^1 = (p_{\{14\}}^{\{1\}} p_{\{1\}}^{\{1\}}) = (0.1642)(0.7877) = 0.1279$

  $p_1^2 = (p_{\{14\}}^{\{2\}} p_{\{1\}}^{\{2\}}) = (0.05)(0.98) = 0.049$

- Agent 2: $p_2^1 = (p_{\{2356\}}^{\{1\}})(p_{\{2\}}^{\{1\}}) = (0.8375)(0.2883) = 0.2414$

  $p_2^2 = (p_{\{2356\}}^{\{2\}})(p_{\{2\}}^{\{2\}}) = (0.95)(0.04) = 0.038$

- Agent 3: $p_3^1 = (p_{\{2356\}}^{\{1\}})(p_{\{356\}}^{\{1\}})(p_{\{3\}}^{\{1\}}) = (0.8375)(0.7117)(0.9959) = 0.5936$

  $p_3^2 = (p_{\{2356\}}^{\{2\}})(p_{\{356\}}^{\{2\}})(p_{\{3\}}^{\{2\}}) = (0.95)(0.96)(0.841) = 0.7669$

- Agent 4: $p_4^1 = (p_{\{14\}}^{\{1\}} p_{\{4\}}^{\{1\}}) = (0.1624)(0.2122) = 0.0344$

  $p_4^2 = (p_{\{14\}}^{\{2\}} p_{\{4\}}^{\{2\}}) = (0.05)(0.02) = 0.001$

- Agent 5: $p_5^1 = (p_{\{2356\}}^{\{1\}})(p_{\{356\}}^{\{1\}})(p_{\{56\}}^{\{1\}})(p_{\{5\}}^{\{1\}}) = (0.8375)(0.7117)(0.04)(0.5339) = 0.1295$

  $p_5^2 = (p_{\{2356\}}^{\{2\}})(p_{\{356\}}^{\{2\}})(p_{\{56\}}^{\{2\}})(p_{\{5\}}^{\{2\}}) = (0.95)(0.96)(0.159)(0.112) = 0.0162$

- Agent 6: $p_6^1 = (p_{\{2356\}}^{\{1\}})(p_{\{356\}}^{\{1\}})(p_{\{56\}}^{\{1\}})(p_{\{6\}}^{\{1\}}) = (0.8375)(0.7117)(0.04)(0.466) = 0.0113$

  $p_6^2 = (p_{\{2356\}}^{\{2\}})(p_{\{356\}}^{\{2\}})(p_{\{56\}}^{\{2\}})(p_{\{6\}}^{\{2\}}) = (0.95)(0.96)(0.159)(0.888) = 0.1287$

- And we illustrate the matrix:

$$[P]^{j\in J_A}_{6\times 2} = \begin{bmatrix} p^1_1 & p^2_1 \\ p^1_2 & p^2_2 \\ p^1_3 & p^2_3 \\ p^1_4 & p^2_4 \\ p^1_5 & p^2_5 \\ p^1_6 & p^2_6 \end{bmatrix} = \begin{bmatrix} 0.1279 & 0.049 \\ 0.2414 & 0.038 \\ 0.5936 & 0.7669 \\ 0.034 & 0.001 \\ 0.1295 & 0.016 \\ 0.011 & 0.1287 \end{bmatrix}$$

- **Step 7.** We use it as well as the $[\Pi^j]^{j\in J_A}_{2\times 1}, [P]^{j\in J_B}_{6\times 6}, [\Pi^j]^{j\in J_B}_{6\times 1}$ as inputs in another Monte Carlo simulation model: $[\Pi_i]_{8\times 1} = [P]^{j\in J_A}_{6\times 2}[\Pi^j]^{j\in J_A}_{2\times 1} + [P]^{j\in J_B}_{6\times 6}[\Pi^j]^{j\in J_B}_{6\times 1}$
- And the simulation gives:

$$\mu_1 = 48191.6, \mu_2 = 51138.1, \mu_3 = 618755.9, \mu_4 = 4060.6, \mu_5 = 12018.3, \mu_6 = 94409.8$$

$$\sigma_1 = 2706.1, \sigma_2 = 2734.2, \sigma_3 = 36420.4, \sigma_4 = 234.4, \sigma_5 = 722.64, \sigma_6 = 5681.1$$

Verifying the Fair division Among agents:

$$\frac{\mu_1}{\mu_2} = \frac{\sigma_1}{\sigma_2} = \frac{U_1}{U_2} = \frac{0.0582}{0.0617}, \frac{\mu_1}{\mu_3} = \frac{\sigma_1}{\sigma_3} = \frac{U_1}{U_3} = \frac{0.0582}{0.7468}, \frac{\mu_1}{\mu_4} = \frac{\sigma_1}{\sigma_4} = \frac{U_1}{U_4} = \frac{0.0582}{0.0049},$$

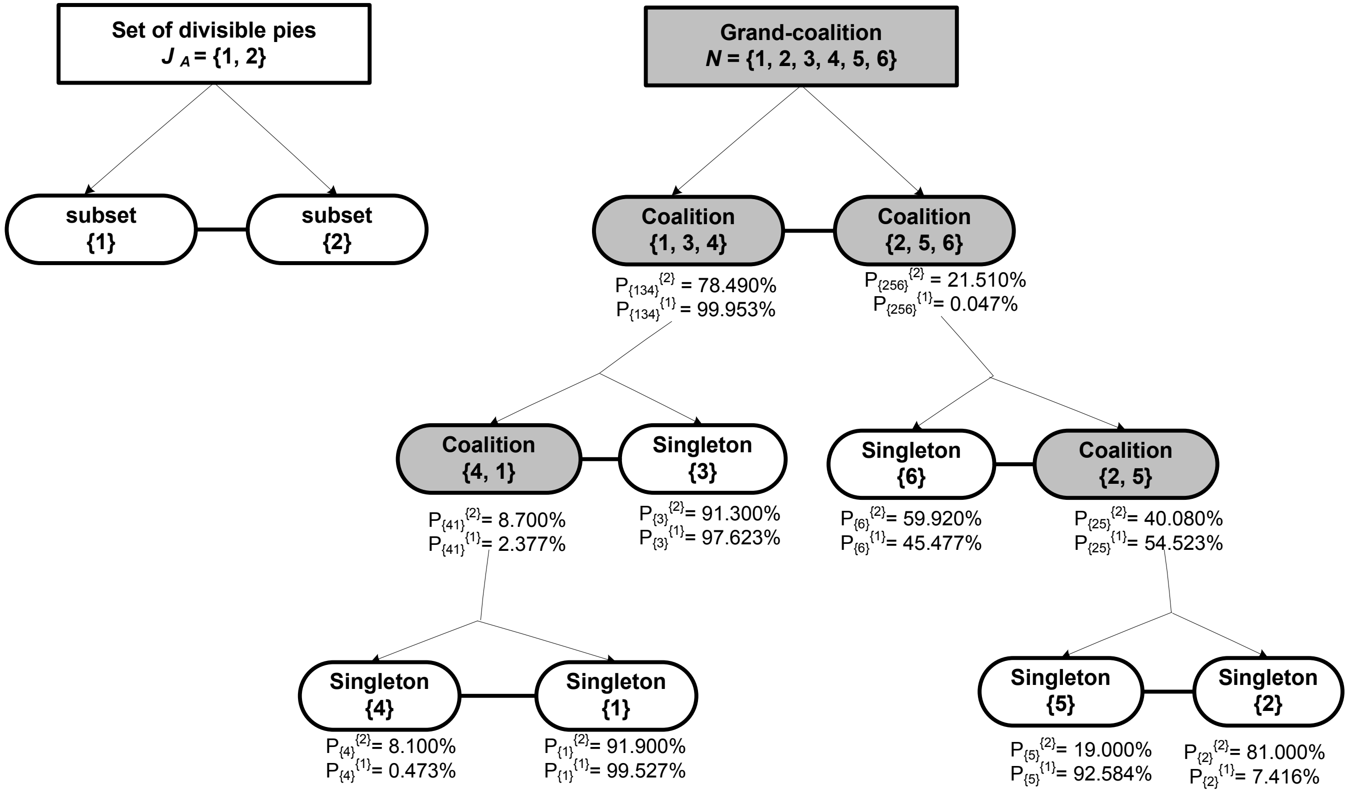
$$\frac{\mu_1}{\mu_5} = \frac{\sigma_1}{\sigma_5} = \frac{U_1}{U_5} = \frac{0.0582}{0.0145}, \frac{\mu_1}{\mu_6} = \frac{\sigma_1}{\sigma_6} = \frac{U_1}{U_6} = \frac{0.0582}{0.1139}, \frac{\mu_2}{\mu_3} = \frac{\sigma_2}{\sigma_3} = \frac{U_2}{U_3} = \frac{0.0617}{0.7468},$$

$$\frac{\mu_2}{\mu_4} = \frac{\sigma_2}{\sigma_4} = \frac{U_2}{U_4} = \frac{0.0617}{0.0049}, \frac{\mu_2}{\mu_5} = \frac{\sigma_2}{\sigma_5} = \frac{U_2}{U_5} = \frac{0.0617}{0.0145}, \frac{\mu_2}{\mu_6} = \frac{\sigma_2}{\sigma_6} = \frac{U_2}{U_6} = \frac{0.0617}{0.1139},$$

$$\frac{\mu_3}{\mu_4} = \frac{\sigma_3}{\sigma_4} = \frac{U_3}{U_4} = \frac{0.7468}{0.0049}, \frac{\mu_3}{\mu_5} = \frac{\sigma_3}{\sigma_5} = \frac{U_3}{U_5} = \frac{0.7468}{0.0145}, \frac{\mu_3}{\mu_6} = \frac{\sigma_3}{\sigma_6} = \frac{U_3}{U_6} = \frac{0.7468}{0.1139},$$

$$\frac{\mu_4}{\mu_5} = \frac{\sigma_4}{\sigma_5} = \frac{U_4}{U_5} = \frac{0.0049}{0.0145}, \frac{\mu_4}{\mu_6} = \frac{\sigma_4}{\sigma_6} = \frac{U_4}{U_6} = \frac{0.0049}{0.1139}, \frac{\mu_5}{\mu_6} = \frac{\sigma_5}{\sigma_6} = \frac{U_5}{U_6} = \frac{0.0145}{0.1139}$$

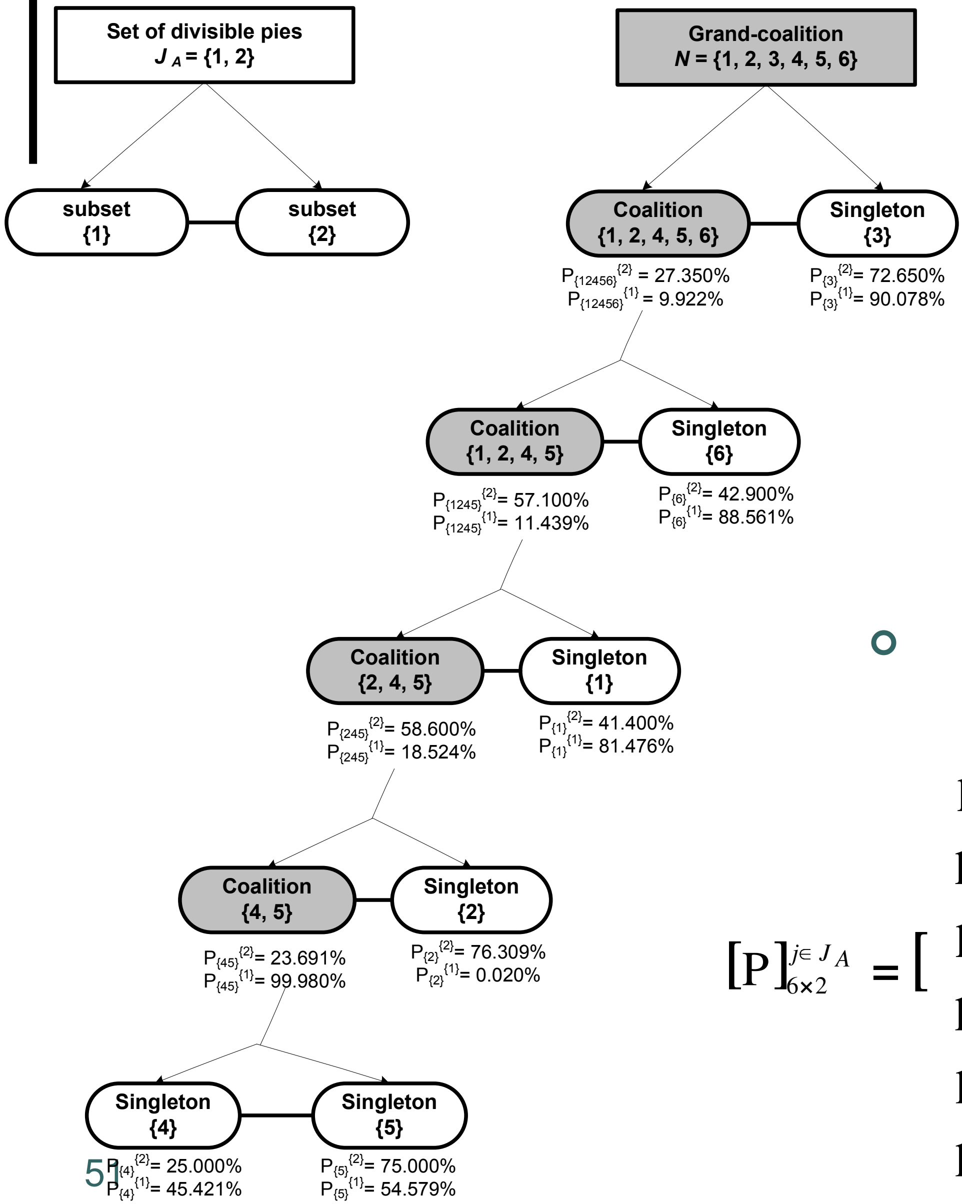
- This matrix is one out of 945 matrices $[P]_{6\times 2}^{j\in J_A}$ that can be computed. For example, if we follow a different set of partitions:



- Then we compute:

$$[P]_{6\times 2}^{j\in J_A} = \begin{bmatrix} p_1^1 & p_1^2 \\ p_2^1 & p_2^2 \\ p_3^1 & p_3^2 \\ p_4^1 & p_4^2 \\ p_5^1 & p_5^2 \\ p_6^1 & p_6^2 \end{bmatrix} = \begin{bmatrix} 0.0236 & 0.0627 \\ 0.0001 & 0.0698 \\ 0.9757 & 0.7166 \\ 0.0001 & 0.0055 \\ 0.0002 & 0.0163 \\ 0.0002 & 0.1288 \end{bmatrix}$$

- … or a different set of partitions:



- Then we compute:

$$[P]_{6\times 2}^{j\in J_A} = \begin{bmatrix} p_1^1 & p_1^2 \\ p_2^1 & p_2^2 \\ p_3^1 & p_3^2 \\ p_4^1 & p_4^2 \\ p_5^1 & p_5^2 \\ p_6^1 & p_6^2 \end{bmatrix} = \begin{bmatrix} 0.0092 & 0.0646 \\ 0.0001 & 0.0698 \\ 0.9008 & 0.7265 \\ 0.0009 & 0.0054 \\ 0.0011 & 0.0162 \\ 0.0878 & 0.1173 \end{bmatrix}$$

Limitations of the model:

- This stochastic model is a static model, rather than a dynamic one, as it does not considers the effect of time.

- Fairness can be achieved if the bargaining outcome is not fixed, as for example in the case study:

$$U = (U_1, U_2, U_3, U_4, U_5, U_6,) = (0.0582, 0.0617, 0.7468, 0.0049, 0.0145, 0.1139)$$

- In particular, this outcome should include the proportional division rule of the final outcomes (Remunerations of the different studies), i.e. those that will be realized. However, the contract among studiers should include a mathematic formula:

$$p_i^1 = p_i^2 = \frac{\Phi_i}{\sum_{i=1}^{n} \Phi_i}, \forall\, i \in N$$

Future research issues:

- Development of a code, e.g. in MatLab for the computation algorithm.
- The application of the proposed method in other real-life applications, such as water-sharing problems, power distribution case, tax-paying problems, etc.
- The application of other solution concepts, e.g. Shapley value, nucleolus in stochastic environments, in which the size of the pie over which agents negotiate is uncertain.
- The development of another model, in which one (or more than one) natural axioms will be used, in order to define a unique solution.
- To examine situations by relaxing the assumptions used in our model: i.e. what happens if pies are indivisible, or/and follow asymmetric probability distributions (e.g. lognormal), if players are risk-averse or/and risk-seeking and are not indifferent between the *m* pies, etc.

THANK YOU!

Athanasios C. Karmperis

………. And some marketing for our recently published Book :